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The equilibrium of a charged elastic sphere

A Nduka
Department of Physics, University of Ife, Ile-Ife, Nigeria

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Abstract. We present a relativistic model for a charged elastic sphere in equilibrium under its own gravitational and electrical repulsion. Equilibrium configurations of such systems imply that in addition to the usual condition of Bonnor a further condition is to be satisfied by the scalars of Rayner.

1. Introduction

Bonnor (1965) has pointed out that a spherical body can remain in equilibrium under a balance of gravitational attraction and electrical repulsion. His solution shows that a spherically symmetric dust cloud of arbitrarily large mass and arbitrarily small radius can remain in equilibrium if the electric charge density and the mass density are equal in magnitude.

In this paper we apply Rayner's theory of elasticity in general relativity to show that if the spherical body is elastic, equilibrium configuration implies that an extra condition must be imposed on the system.

2. Solutions of the field equations

The appropriate line element for a static spherically symmetric system is

$$ds^2 = f^{-2}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + f^2\, dt^2 \tag{2.1}$$

where $f(r)$ is a real function. The four-dimensional velocity vector $u^i = dx^i/ds$ is assumed to have $u^1 = u^2 = u^3 = 0$ since the sphere is static. It then follows from equation (2.1) that

$$(u^0)^2 = f^{-2}. \tag{2.2}$$

The fluid is described by a density ρ, a four-velocity $u^i = (u^0, 0, 0, 0)$, an elasticity tensor C_{ijkl} and an electric current $J^i = (J^0, 0, 0, 0)$. In this paper we shall assume that the pressure vanishes. Then the stress-energy tensor $T^i{}_k$ is given by

$$T^i{}_k = \tau^i{}_k + \epsilon^i{}_k. \tag{2.3}$$

Here $\tau^i{}_k$ is the energy–momentum tensor for the elastic matter (Rayner 1963):

$$\tau^i{}_k = \rho u^i u_k - \tfrac{1}{2} g^{il} C^{rs}_{kl}(\tilde{g}_{rs} - \tilde{g}^0_{rs}) \tag{2.4}$$

where the tensor C_{ijkl} admits the representation

$$C_{ijkl} = \nu(r)\tilde{g}^0_{ij}\tilde{g}^0_{kl} + \mu(r)(\tilde{g}^0_{ik}\tilde{g}^0_{jl} + \tilde{g}^0_{il}\tilde{g}^0_{jk}), \tag{2.5}$$

$\tilde{g}_{ij} = g_{ij}+u_i u_j$ is the metric for the deformed elastic body and $\tilde{g}^0_{ij}$ is the metric for the undeformed elastic body; $\tilde{g}^0_{ij}$ is taken as flat and the only non-vanishing components are (Roy and Singh 1973):

$$\tilde{g}^0_{11} = -1, \qquad \tilde{g}^0_{22} = -r^2, \qquad \tilde{g}^0_{33} = -r^2 \sin^2\theta \tag{2.6}$$

and $\epsilon^i{}_k$ is the electromagnetic energy tensor:

$$\epsilon^i{}_k = \frac{1}{4\pi}(g^{rs}F^i{}_r F_{ks} - \tfrac{1}{4}\delta^i{}_k F_{rs}F^{rs}). \tag{2.7}$$

Here F^{ik} is the Minkowski electromagnetic field tensor:

$$F_{ik} = A_{i,k} - A_{k,i} \tag{2.8}$$

where A_i is the four-potential.

The only non-vanishing components of C_{ijkl} are C_{1111}, C_{1122}, C_{1133}, C_{2222}, C_{2233}, C_{3333} with the values

$$\begin{aligned} C_{1111} &= \nu+2\mu, \qquad C_{1122} = \nu r^2, \qquad C_{1133} = \nu r^2 \sin^2\theta, \\ C_{2222} &= (\nu+2\mu)r^4, \qquad C_{2233} = \nu r^4 \sin^2\theta, \qquad C_{3333} = (\nu+2\mu)r^4 \sin^4\theta. \end{aligned} \tag{2.9}$$

Spherical symmetry requires only the radial component of the electric field, $F^{01} = -F^{10}$, to be non-vanishing. This implies that $A_0 = \alpha(r)$, and $A_1 = A_2 = A_3 = 0$. Then from equations (2.7) and (2.8) we have the nonzero components of $\epsilon^i{}_k$:

$$\epsilon^0{}_0 = \epsilon^1{}_1 = -\epsilon^2{}_2 = -\epsilon^3{}_3 = \alpha'^2/8\pi \tag{2.10}$$

where a prime denotes differentiation with respect to r. From equations (2.1), (2.2), (2.4) and (2.9) we have the nonzero components of $\tau^i{}_k$:

$$\tau^0{}_0 = \rho \tag{2.11}$$

$$\tau^1{}_1 = -\tfrac{1}{2}(1+f^2)(3\nu+2\mu) \tag{2.12}$$

$$\tau^2{}_2 = -\tfrac{1}{2}(1+f^2)(3\nu+2\mu) \tag{2.13}$$

$$\tau^3{}_3 = -\tfrac{1}{2}(1+f^2)(3\nu+2\mu). \tag{2.14}$$

The Einstein–Maxwell field equations for the charged elastic matter are (Rayner 1963, Adler *et al* 1965):

$$G^i{}_k = -8\pi(\tau^i{}_k + \epsilon^i{}_k) \tag{2.15}$$

$$[(-g)^{1/2}F^{ik}]_{,k} = 4\pi J^i(-g)^{1/2} \tag{2.16}$$

$$F_{[ik,l]} = 0. \tag{2.17}$$

Here $G^i{}_k$ is the Einstein tensor and g is the determinant of the metric tensor. In this paper we work in relativistic units throughout ($c = 1$ and $K = 1$; K is the gravitational constant).

Then from equations (2.1), (2.10)–(2.15) we obtain the field equations:

$$-G^0{}_0 = 2ff'' - 3f'^2 + 4ff'/r = 8\pi\rho + \alpha'^2 \tag{2.18}$$

$$G^1{}_1 = -f'^2 = 4\pi(1+f^2)(3\nu+2\mu) - \alpha'^2 \tag{2.19}$$

$$G^2{}_2 = G^3{}_3 = 4\pi(1+f^2)(3\nu+2\mu) + \alpha'^2 = f'^2 \tag{2.20}$$

from which we conclude that

$$2ff'' - 3f'^2 + 4ff'/r = 8\pi\rho + \alpha'^2 \tag{2.21}$$

$$f'^2 = \alpha'^2 - 4\pi(1+f^2)(3\nu+2\mu) \tag{2.22}$$

$$f'^2 = \alpha'^2 + 4\pi(1+f^2)(3\nu+2\mu). \tag{2.23}$$

As noted earlier only $F^{01} = -F^{10}$ is non-vanishing. This case satisfies equation (2.17) whereas equation (2.16) shows that $F^{01} = F^{01}(r)$. Then by equation (2.8) we conclude that $A_0 = \alpha(r)$, in agreement with our earlier observation.

Equations (2.22) and (2.23) imply that

$$(1+f^2)(3\nu+2\mu) = 0.$$

But since f is a real function it follows that $(1+f^2) \neq 0$, so that $(3\nu+2\mu) = 0$, or

$$\mu = -\tfrac{3}{2}\nu. \tag{2.24}$$

Then equations (2.21)–(2.23) become

$$2ff'' - 3f'^2 + 4ff'/r = 8\pi\rho + \alpha'^2 \tag{2.25}$$

$$f'^2 = \alpha'^2 \tag{2.26}$$

$$f'^2 = \alpha'^2. \tag{2.27}$$

Bonnor (1965) obtained these same equations, (2.25)–(2.27), for the case of a spherically symmetric dust cloud. He showed that such a system can remain in equilibrium if the electric charge density σ and the mass density ρ satisfy the condition

$$\sigma = \pm\rho. \tag{2.28}$$

It then follows that a spherically symmetric elastic dust cloud can remain in equilibrium if both the conditions (2.24) and (2.28) are satisfied.

3. Conclusion

Our solution shows that a spherically symmetric elastic matter fluid can remain in equilibrium if the conditions (2.24) and (2.28) are satisfied.

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